

Discrete Flavour Symmetries from the Heisenberg group

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Abstract

Non-abelian discrete symmetries are of particular importance in model building. They are mainly invoked to explain the various fermion mass hierarchies and forbid dangerous superpotential terms. In string models they are usually associated to the geometry of the compactification manifold and more particularly to the magnetised branes in toroidal compactifications. Motivated by these facts, in this note we propose a unified framework to construct representations of finite discrete family groups based on the automorphisms of the discrete and finite Heisenberg group. We focus in particular, in the $PSL_2(p)$ groups which contain the phenomenologically interesting cases.

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1 Introduction

Non-abelian discrete symmetries play a prominent rôle in model building. Among other objectives, more than a decade ago, they have been widely used to interpret the neutrino data in various extensions of the Standard Model [1, 2, 3]. The ensuing years there were attempts to construct them in the context of string theory models. Within this framework, important activity has been focused on elaborating predictions for physically measurable quantities such as mass textures and CP-violating matrices. Indeed, (non-abelian) discrete symmetries in string theory emerge in the context of various compactifications and recently they have attracted considerable attention [4]–[10]. In fact, it has been realised that they can act as family symmetries which restrict the arbitrary Yukawa parameters of the superpotential and lead to acceptable quark and lepton masses and mixing. Moreover, they can suppress undesired proton decay operators and various -yet unobserved- flavour violating interactions.

More recently, the implementation of the idea of discrete symmetries has also been considered in F-theory constructions. In F-theory [11] the elliptically fibred space consists of a $K3$ manifold with a torus attached at each point. The τ -modulus of the torus is defined in terms of the two scalar fields of the type IIB string theory and the fibration is described by the Weierstraß model. According to the standard interpretation, the associated geometric singularities (classified as ADE types) are linked to the gauge symmetries of the effective models. The highest singularity of the elliptic fibration is described by the E_8 exceptional symmetry, so that ordinary successful GUT symmetries such as $SU(5)$ and $SO(10)$ are easily embedded [12] in the maximal group E_8 and correspond to a particular divisor of the internal manifold. Hence, the remaining symmetry can in principle accommodate some suitable non-abelian discrete group which could act as a family symmetry. If, for example, the GUT model is $SU(5)$ which is the minimal symmetry accommodating the Standard Model, then the commutant with respect to E_8 is also $SU(5)$ (denoted usually as perpendicular, $SU(5)_\perp$, to the GUT). The latter naturally incorporates phenomenologically viable non-abelian discrete groups [9], such as S_n, A_n where usually $n \leq 5$ and more generally $PSL_2(p)$, where $p \leq 11$.

The last couple of years, in the context of F-theory, several works focused also in the low energy implications of the two torus geometry, in a different approach. In general, discrete symmetries in these constructions are of Abelian nature. Such cases are the Torsion part of the Mordell-Weil group of rational points on elliptic curves and more generally the Tate-Shafarevich group which has been shown to determine the discrete symmetries arising in F-theory [6].

Furthermore, there exist cases in string theory [4] where non-abelian finite groups may

emerge as well. In this context a class of non-Abelian discrete symmetries may arise from discrete isometries of the torus geometry, on which the Heiseberg group has a natural action. Discrete non-abelian symmetries are also realised in magnetised D-brane models in toroidal compactifications [4, 8]. Along these lines, explicit F-theory constructions have also recently appeared [10]. Let us finally note that finite groups as subgroups of continuous non-abelian symmetries have been discussed and classified in an orbifold context [13].

Motivated by the above facts, in the present work we will develop a unified method for the construction of the smaller non-trivial representations of certain finite groups. Because their main rôle in particle physics is to discriminate the three fermion families we are focusing mainly on the $PSL_2(p)$ groups possessing triplet representations.

2 The non-abelian discrete groups $SL_2(p)$ and $PSL_2(p)$

Among the various discrete symmetries used to interpret the fermion mass hierarchy are the special linear groups $SL_2(p)$ [14, 15] and their corresponding projective restrictions $PSL_2(p)$. Some of these groups coincide with the symmetries of regular polyedra in three or higher dimensional space dimensions.

Not all the representations of these groups are relevant for model building. Because only three families of fermions exist in nature, only the groups with particular representations related to Yukawa and gauge couplings are considered in the literature [1, 2, 3].

Most of the phenomenologically viable cases, are included in the projective linear groups $PSL_2(p)$ for $p = 3, 5, 7$ and 11. All of them support triplets which are suitable representations to accommodate the three fermion generations. More specifically, for $p = 3$ we obtain $PSL_2(3)$ which is isomorphic to the alternating group A_4 . Furthermore, the group $PSL_2(5)$ is isomorphic to the smallest non-abelian simple group A_5 . The case of $PSL_2(7)$ is also phenomenologically interesting [16, 17, 18].

The $PSL_2(p)$ groups for values $p \leq 11$ can be naturally incorporated in an F-theory context. Indeed, for the most common grand unified (GUT) models such as $SU(5)$, $SO(10)$, E_6 embedded in the E_8 singularity, the possible gauge groups which could act as family symmetries can be read off from the embedding formula

$$E_8 \supset \frac{E_n \times SU(m)}{Z_m}, \quad n + m = 9 \quad (2.1)$$

Therefore, we have the following cases

$$E_8 \supset E_6 \times SU(3)_\perp \quad (2.2)$$

$$E_8 \supset E_5 \times SU(4)_\perp \cong SO(10) \times SU(4)_\perp \quad (2.3)$$

$$E_8 \supset SU(5) \times SU(5)_\perp \quad (2.4)$$

From the above, we see that the corresponding flavour discrete groups should be embedded in $SU(3)_\perp$, $SU(4)_\perp$ and $SU(5)_\perp$. Indeed, $A_{4,5}$ are subgroups of $SO(3) \sim SU(2)$, $PSL_2(7)$ is a subgroup of $SU(3)$ and $PSL_2(11)$ is contained in $SU(5)$.

In this work, we present a unified approach for constructing the explicit relevant representations of these groups. The reason is that in the literature, up to our knowledge, while only explicit ad-hoc constructions have been presented, a systematic use of the theory of the representation of these particular groups does not exist yet.

3 Definition of $SL_2(p)$ and $PSL_2(p)$ groups

The $SL_2(p)$ group is defined in the simplest way as a group of 2×2 matrices with elements integers modulo p , where p is a prime integer, and determinant one modulo p . These groups usually are generated by two elements which obey certain conditions and these define what is called a presentation of the group. These conditions depend on p but there is a universal presentation given by two generators which are conjugate one to another under the matrix

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

These universal elements (for any value of p) are the following two matrices

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

We can readily observe that each one of them generate an abelian group of order p and that they satisfy the Braid relation

$$RLR = LRL$$

In the literature another basis of generators is used which is called Artin's presentation. These are defined through the Braid generators, as

$$a = RLR, \quad b = RL$$

and so they are

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad (3.1)$$

They satisfy the relations

$$a^2 = b^3 = -I \quad (3.2)$$

We can invert the relation of the two sets of generators as

$$R = b^{-1}a, \quad L = a^{-1}b^2 \quad (3.3)$$

Apart from equations (3.2), the presentation in terms of a, b contains additional relations depending on the value of p .

The group $SL_2(p)$ has a normal subgroup of order two $Z_2 = \{1, -1\}$, so the coset space $SL_2(p)/\{1, -1\}$ is a group which is called the projective group $PSL_2(p)$.

Our aim is to construct some basic non-trivial irreducible representations of $PSL_2(p)$ out of which all others -physically relevant- are generated by tensor products. To this end, we are going to use a particular representation of $SL_2(p)$ which is known as the Weil metaplectic representation. In the physics literature it has been introduced by the work of Balian and Itzykson [19]. Many details of this particular unitary representation with various applications has been presented in [20, 21]. As it will be shown in the next section, this representation is reducible in two irreducible unitary representations of dimensions $\frac{p+1}{2}$ and $\frac{p-1}{2}$. Thus, we obtain discrete subgroups of the unitary groups $SU(\frac{p\pm 1}{2})$. For example, when $p = 3$ we obtain discrete subgroups of $SU(2)$ and $U(1)$, for $p = 5$ we get discrete subgroups of $SU(3)$ and $SU(2)$ and so on.

4 Heiseberg-Weyl group HW_p and the metaplectic representation of $SL_2(p)$

The Finite Heisenberg group HW_p [22], is defined as the set of 3×3 matrices of the form

$$g(r, s, t) = \begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ t & s & 1 \end{pmatrix} \quad (4.1)$$

where r, s, t belong to \mathbb{Z}_p (integers modulo p), where the multiplication of two elements is carried modulo p .

When p is a prime integer there is a unique p -dimensional unitary irreducible and faithful representation of this group, given by the following matrices

$$J_{r,s,t} = \omega^t P^r Q^s \quad (4.2)$$

where $\omega = e^{2\pi i/p}$, i.e. the p^{th} primitive root of unity and the matrices P, Q are defined as

$$P_{kl} = \delta_{k-1,l} \quad (4.3)$$

$$Q_{kl} = \omega^k \delta_{kl} \quad (4.4)$$

where $k, l = 0, \dots, p-1$.

It is to be observed that, if ω is replaced with ω^k , for $k = 1, 2, \dots, p-1$ all the relations above remain intact. Since p is prime all the resulting representations are p -dimensional and inequivalent.

The matrices P, Q satisfy the fundamental Heisenberg commutation relation of Quantum Mechanics in an exponentiated form

$$Q P = \omega P Q \quad (4.5)$$

In the above, Q represents the position operator on the circle \mathbb{Z}_p of the p roots of unity and P the corresponding momentum operator. These two operators are related by the diagonalising unitary matrix F of P ,

$$Q F = F P \quad (4.6)$$

so F is the celebrated Discrete Fourier Transform matrix

$$F_{kl} = \frac{1}{\sqrt{p}} \omega^{kl}, \text{ with } k, l = 0, \dots, p-1 \quad (4.7)$$

An important subset of HW_p consists of the magnetic translations

$$J_{r,s} = \omega^{rs/2} P^r Q^s \quad (4.8)$$

with $r, s = 0, \dots, p-1$. These matrices are unitary ($J_{r,s}^\dagger = J_{-r,-s}$) and traceless, and they form a basis for the Lie algebra of $SL(p, \mathbb{C})$. They satisfy the important relation

$$J_{r,s} J_{r',s'} = \omega^{(r's - rs')/2} J_{r+r', s+s'} \quad (4.9)$$

This relation implies that the magnetic translations form a projective representation of the translation group $\mathbb{Z}_p \times \mathbb{Z}_p$. The factor of $1/2$ in the exponent of (4.9) must be taken modulo p .

The $SL_2(p)$ appears here as the automorphism group of magnetic translations and this defines the Weil's metaplectic representation. If we consider the action of an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the coordinates (r, s) of the periodic torus $\mathbb{Z}_p \times \mathbb{Z}_p$, this induces a unitary automorphism $U(A)$ on the magnetic translations, since the representation of Heisenberg group is unitary and irreducible,

$$U(A) J_{r,s} U^\dagger(A) = J_{r',s'}$$

where (r', s') are given by

$$(r', s') = (r, s) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (4.10)$$

This relation determines $U(A)$ up to a phase and in the case of $A \in SL_2(p)$, the phase can be fixed to give an exact (and not projective) unitary representation of $SL_2(p)$.

The detailed formula of $U(A)$ has been given by Balian and Itzykson [19]. Depending on the specific values of the a, b, c, d parameters of the matrix A , we distinguish the following cases:

$$\delta \neq 0 : \quad U(A) = \frac{\sigma(1)\sigma(\delta)}{p} \sum_{r,s} \omega^{[br^2+(d-a)rs-cs^2]/(2\delta)} J_{r,s} \quad (4.11)$$

$$\delta = 0, b \neq 0 : \quad U(A) = \frac{\sigma(-2b)}{\sqrt{p}} \sum_s \omega^{s^2/(2b)} J_{s(a-1)/b,s} \quad (4.12)$$

$$\delta = b = 0, c \neq 0 : \quad U(A) = \frac{\sigma(2c)}{\sqrt{p}} \sum_r \omega^{-r^2/(2c)} P^r \quad (4.13)$$

$$\delta = b = 0 = c = 0 : \quad U(1) = I \quad (4.14)$$

where $\delta = 2 - a - d$ and $\sigma(a)$ is the Quadratic Gauss sum given by

$$\sigma(a) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \omega^{ak^2} = (a|p) \times \begin{cases} 1 & \text{for } p = 4k + 1 \\ i & \text{for } p = 4k - 1 \end{cases} \quad (4.15)$$

while the Legendre symbol takes the values $(a|p) = \pm 1$ depending whether a is or is not a square modulo p .

It is possible to perform explicitly the above Gaussian sums noticing that

$$(J_{r,s})_{k,l} = \delta_{r,k-l} \omega^{\frac{k+l}{2}s} \quad (4.16)$$

where all indices take the values $k, l, r, s = 0, \dots, p-1$. This has been done in [20, 21]. In the case $\delta = 2 - a - d \neq 0$ and $c \neq 0$, the result is

$$\delta \neq 0 : \quad U(A)_{k,l} = \frac{(-2c|p)}{\sqrt{p}} \times \begin{Bmatrix} 1 \\ -i \end{Bmatrix} \omega^{-\frac{ak^2-2kl+dl^2}{2c}} \quad (4.17)$$

If $c = 0$, then we transform the matrix A to one with $c \neq 0$. The other cases $\delta = 0$ can be worked out easily using the matrix elements of $J_{r,s}$ given in (4.16).

It is interesting to notice that redefining ω to become ω^k for $k = 1, 2, \dots, p-1$, the matrix $U(A)$ transforms to the matrix $U(A_k)$, where A_k is the 2×2 matrix $A_k = \begin{pmatrix} a & bk \\ c/k & d \end{pmatrix}$, which belongs to the same conjugacy class with A as long as k is a quadratic residue. If $k = p-1$ we pass from the representation $U(A)$ to the complex conjugate one $U(A)^*$.

The Weyl representation presented above, provides the interesting result that the unitary matrix corresponding to the $SL_2(p)$ element $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is -up to a phase- the Discrete Finite Fourier Transform (4.7)

$$U(a) = (-1)^{k+1} i^n F$$

where $n = 0$ for $p = 4k + 1$ and $n = 1$ for $p = 4k - 1$.

The Fourier Transform matrix generates a fourth order abelian group with elements

$$F, F^2 = S, F^3 = F^*, F^4 = I \quad (4.18)$$

The matrix S represents the element $a^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Its matrix elements are

$$S_{k,l} = \delta_{k,-l}, \quad k, l = 0, \dots, p-1 \quad (4.19)$$

$$U(a^2)_{k,l} = i^{2n} S_{k,l} = (-1)^n \delta_{k,-l}, \quad k, l = 0, \dots, p-1 \quad (4.20)$$

Because the action of S on $J_{r,s}$ changes the signs of r, s , while $\forall A \in SL_2(p)$ the unitary matrix $U(A)$ depends quadratically on r, s in the sum (4.11), it turns out that S commutes with all $U(A)$. Moreover, $S^2 = I$ and we can construct two projectors

$$P_+ = \frac{1}{2}(I + S), \quad P_- = \frac{1}{2}(I - S)$$

with dimensions of their invariant subspaces $\frac{p+1}{2}$ and $\frac{p-1}{2}$ correspondingly. So the Weil p -dimensional representation is the direct sum of two irreducible unitary representations

$$U_+(A) = U(A)P_+, \quad U_-(A) = U(A)P_-$$

To obtain the block diagonal form of the above matrices $U_{\pm}(A)$, we rotate with the orthogonal matrix of the eigenvectors of S . This p -dimensional orthogonal matrix, dubbed here O_p , can be obtained in a maximally symmetric form (along the diagonal as well as along the anti-diagonal) using the eigenvectors of S in the following order: In the first $(p+1)/2$ columns we put the eigenvectors of S of eigenvalue equal to 1, and in the next $(p-1)/2$ columns the eigenvectors of eigenvalue equal to -1 in the specific order given below:

$$(e_0)_k = \delta_{k0}, \quad (4.21)$$

$$(e_j^+)_k = \frac{1}{\sqrt{2}}(\delta_{k,j} + \delta_{k,-j}), \quad j = 1, \dots, \frac{p-1}{2} \quad (4.22)$$

$$(e_j^-)_k = \frac{1}{\sqrt{2}}(\delta_{k,j} - \delta_{k,-j}), \quad j = \frac{p+1}{2}, \dots, p \quad (4.23)$$

where $k = 0, \dots, p-1$.

Different orderings of eigenvectors may lead to different forms of the matrices $U_{\pm}(A)$. The so obtained orthogonal matrix O_p has the property

$$O_p^2 = I$$

due to its symmetric form.

The final block diagonal form of $U_{\pm}(A)$ is obtained through an O_p rotation

$$V_{\pm}(A) = O_p U(A)_{\pm} O_p \quad (4.24)$$

5 The construction of the $SL_2(p)$ generators

In this section we are going to give explicit expressions for the $SL_2(p)$ generators a, b for any value of p in the $\frac{p \pm 1}{2}$ irreducible representations. We will also consider the corresponding matrix expressions of the projective group $PSL_2(p)$.

According to the above construction the two generators a, b have the following unitary matrix representations

$$U_{\pm}(a) = (-1)^{k+1} i^n F \frac{I \pm S}{2} = (-1)^{k+1} i^n \frac{1}{2} (F \pm F^*) \quad (5.1)$$

with matrix elements

$$[U_{\pm}(a)]_{k,l} = (-1)^{k+1} i^n \frac{1}{2\sqrt{p}} (\omega^{kl} \pm \omega^{-kl})$$

where, as noted previously, $n = 0$ for $p = 4k + 1$ and $n = 1$ for $p = 4k - 1$.

In order to bring this in the block diagonal form (4.24) we need to perform a rotation with O_p :

$$A^{[\frac{p \pm 1}{2}]} = O_p U_{\pm}(a) O_p. \quad (5.2)$$

For the second generator, b , given in (3.1) we obtain

$$U(b)_{k,l} = \frac{1}{\sqrt{p}} (-1)^{\frac{p^2-1}{8}} \begin{Bmatrix} 1 \\ i \end{Bmatrix} \omega^{-\frac{l^2}{2} + kl} \quad (5.3)$$

and so,

$$U_{\pm}(b)_{k,l} = \frac{1}{2\sqrt{p}} (-1)^{\frac{p^2-1}{8}} \begin{Bmatrix} 1 \\ i \end{Bmatrix} (\omega^{-\frac{k^2}{2} + kl} \pm \omega^{-\frac{l^2}{2} - kl})$$

where $k, l = 0, \dots, p-1$. As previously, in order to get the block diagonal form we have to rotate the so obtained matrix with O_p :

$$B^{[\frac{p \pm 1}{2}]} = O_p U_{\pm}(b) O_p. \quad (5.4)$$

Our final goal is to obtain some basic representations of $PSL_2(p)$ which will be used to build higher dimensional ones. We observe that the difference between $SL_2(p)$ and $PSL_2(p)$ in the defining relations of generators a and b is that, for $SL_2(p)$ one has to take $a^2 = b^3 = -I$ while for $PSL_2(p)$ we have the relations $a^2 = b^3 = I$. This last requirement comes from the different action of $SL_2(p)$ and $PSL_2(p)$ which are linear and Möbius correspondingly.

We can obtain irreducible representations of $PSL_2(p)$ from irreducible representations of $SL_2(p)$ in the following way. Taking into consideration the above observation we must

find the representations of $SL_2(p)$ for which $(A^{[\frac{p+1}{2}]})^2 = (B^{[\frac{p+1}{2}]})^3 = I$. We can easily check that this happens for the $\frac{p+1}{2}$ dimensional representation only when $p = 4k + 1$, and for the $\frac{p-1}{2}$ one only when $p = 4k - 1$. This way, we get $(2k + 1)$ and $(2k - 1)$ -dimensional irreps of $PSL_2(p)$ correspondingly.

6 Examples: The cases $p = 3, 5, 7$

In this section using the method described above, we will present examples, considering the cases $p = 3, 5$ and $p = 7$. It is straightforward to construct similar representations of higher values of p .

6.1 The case $p = 3$

The resulting group is $SL_2(3)$ which has 24 elements, while its projective subgroup $PSL_2(3)$ has 12 elements and is isomorphic to A_4 , the symmetry group of the even permutations of four objects,³ or the symmetry group of the Tetrahedron T . The symmetry groups of the Cube and the Octahedron is S_4 which is isomorphic to $PGL_2(3)$, the automorphism group of $SL_2(3)$.

The generators in the doublet representation are the following: the $A^{[2]}$ -representation is

$$A^{[2]} = -\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & \eta + \eta^2 \end{pmatrix} = -\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \quad (6.1)$$

where we have used the fact that $1 + \eta + \eta^2 = 0$. The representation $B^{[2]}$ is given by

$$B^{[2]} = -\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2}\eta \\ \sqrt{2} & 1 + \eta^2 \end{pmatrix} = -\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2}\eta \\ \sqrt{2} & -\eta \end{pmatrix}. \quad (6.2)$$

They satisfy the $SL_2(3)$ relations

$$A^{[2]2} = B^{[2]3} = (A^{[2]}B^{[2]})^3 = -I \quad (6.3)$$

The singlet representations are

$$A^{[1]} = -\frac{i}{\sqrt{3}}(\eta - \eta^2) \quad (6.4)$$

$$B^{[1]} = -\frac{i}{\sqrt{3}}(\eta^2 - 1) \quad (6.5)$$

³ A_4 is suitable to reproduce the TriBi-maximal mixing to leading order in the neutrino sector and has been discussed in many works including [23]. See reviews [1,2,3] for a complete list of related papers.

The defining relations are satisfied

$$A^{[1]^2} = B^{[1]^3} = (A^{[1]}B^{[1]})^3 = 1 \quad (6.6)$$

and are consistent with the $PSL_2(3) \sim A_4$. From (6.4,6.5) we deduce

$$A^{[1]} \cdot B^{[1]} \equiv \eta, \quad (A^{[1]} \cdot B^{[1]})^2 \equiv \eta^2$$

so that we can define the singlet representations with the standard multiplications rules:

$$1' : s_{1'} = \eta, \quad (6.7)$$

$$1'' : s_{1''} = \eta^2, \quad (6.8)$$

$$1' \times 1'' = 1 : s_1 = 1 \quad (6.9)$$

6.2 The cases $p = 5$

Next we elaborate the case of $PSL_2(5)$ which is isomorphic to the symmetry group I of the Dodecahedron and Icosahedron as well as to A_5 . The group $SL_2(5)$ is isomorphic to the symmetry group $2I$ of the Binary Icosahedron. The 60 elements of A_5 are generated by two generators a, b with the properties

$$a^2 = b^3 = (ab)^5 = I$$

With the above method we find two representations of $SL_2(5)$, one of three and a second one of two dimensions.

The first generator is a unitary 3×3 matrix

$$A^{[3]} = -\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \eta + \eta^4 & \eta^2 + \eta^3 \\ \sqrt{2} & \eta^2 + \eta^3 & \eta + \eta^4 \end{pmatrix} = -\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \frac{\sqrt{5}-1}{2} & -\frac{\sqrt{5}+1}{2} \\ \sqrt{2} & -\frac{\sqrt{5}+1}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix} \quad (6.10)$$

where in the last form, the matrix elements have been written in terms of the golden ratio, since

$$\eta + \eta^4 = \frac{1}{2}(\sqrt{5} - 1), \quad \eta^2 + \eta^3 = -\frac{1}{2}(\sqrt{5} + 1)$$

The character of the representation is $\text{Tr} A^{[3]} = -1$, as expected from the character table of $PSL_2(5)$.

The second generator has the following three dimensional representation

$$B^{[3]} = -\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2}\eta^2 & \sqrt{2}\eta^3 \\ \sqrt{2} & \eta^3 + \eta & \eta + 1 \\ \sqrt{2} & \eta^4 + 1 & \eta^4 + \eta^2 \end{pmatrix} \quad (6.11)$$

while the character is $\text{Tr} B^{[3]} \propto 1 + \eta + \eta^2 + \eta^3 + \eta^4 = 0$. It can be readily checked that $A^{[3]}$ and $B^{[3]}$ satisfy the defining relations of the $PLS_2(5)$ group:

$$A^{[3]2} = B^{[3]3} = (A^{[3]} \cdot B^{[3]})^5 = I$$

These generators correspond to $3'$ triplet. Indeed, in order to make contact with the form of generators given in recent literature we transform the above in the s'_5, t'_5 basis ⁴, setting

$$s'_5 \equiv A^{[3]}, \quad t'_5 \equiv A^{[3]} \cdot B^{[3]} \rightarrow B^{[3]} = s'_5 \cdot t'_5$$

Hence, the two new generators s'_5, t'_5 are

$$s'_5 = -\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \eta + \eta^4 & \eta^2 + \eta^3 \\ \sqrt{2} & \eta^2 + \eta^3 & \eta + \eta^4 \end{pmatrix}, \quad t'_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^3 \end{pmatrix} \quad (6.12)$$

They satisfy the defining relations

$$s'^2_5 = t'^5_5 = (s'_5 \cdot t'_5)^3 = I$$

while their characters are

$$\chi_{s'_5} = -1, \quad \chi_{t'_5} = \frac{1 - \sqrt{5}}{2}$$

It is possible to get the other triplet representation of $SL_2(5)$ (up to equivalence) given in [24],

$$s_5 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \eta^2 + \eta^3 & \eta + \eta^4 \\ -\sqrt{2} & \eta + \eta^4 & \eta^2 + \eta^3 \end{pmatrix}, \quad t_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^3 \end{pmatrix} \quad (6.13)$$

by redefining η to η^3 in eq.(6.12). As discussed in section 4 after eq.(4.17) this is equivalent to a rescaling of appropriate elements of $SL_2(5)$ which give the generators s'_5 and t'_5 .

The two-dimensional representation gives

$$A^{[2]} = -\frac{1}{\sqrt{5}} \begin{pmatrix} \eta^4 - \eta & \eta^2 - \eta^3 \\ \eta^2 - \eta^3 & \eta - \eta^4 \end{pmatrix} \quad (6.14)$$

and

$$B^{[2]} = -\frac{1}{\sqrt{5}} \begin{pmatrix} \eta^2 - \eta^4 & \eta^4 - 1 \\ 1 - \eta & \eta^3 - \eta \end{pmatrix} \quad (6.15)$$

satisfying

$$A^{[2]2} = B^{[2]3} = (A^{[2]} \cdot B^{[2]})^5 = -I$$

⁴See for example neutrino models with A_5 family symmetry in [24, 25].

Obviously, the above two-dimensional matrices are representations of $SL_2(5)$, but not of $PSL_2(5)$.

The three and the two-dimensional representations of the generators constructed above, are unitary matrices and so they generate discrete subgroups of $SU(3)$ and $SU(2)$ Lie groups.

6.3 The cases $p = 7$

As a final example in this note, we consider the case $p = 7$. The associated groups are $SL_2(7)$ with 336 elements and its projective one $PSL_2(7)$ which has 168 elements and it is a discrete simple subgroup of $SU(3)$. It is the group preserving the discrete projective geometry of the Fano plane realising the multiplication structure of the octonionic units.

Using the method described above we will construct the four- and three-dimensional representations of $SL_2(7)$. The three-dimensional one is also a representation of $PSL_2(7)$

$$a^2 = b^3 = (ab)^7 = [a, b]^4 = 1$$

with $[a, b] = a^{-1}b^{-1}ab$.

The $A^{[4]}$ and $B^{[4]}$ generating matrices of the irreducible four-dimensional unitary representation of $SL_2(7)$ are

$$A^{[4]} = \frac{i}{\sqrt{7}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \eta + \eta^6 & \eta^2 + \eta^5 & \eta^3 + \eta^4 \\ \sqrt{2} & \eta^2 + \eta^5 & \eta^3 + \eta^4 & \eta + \eta^6 \\ \sqrt{2} & \eta^3 + \eta^4 & \eta + \eta^6 & \eta^2 + \eta^5 \end{pmatrix} \quad (6.16)$$

and

$$B^{[4]} = \frac{i}{\sqrt{7}} \begin{pmatrix} 1 & \sqrt{2}\eta^3 & \sqrt{2}\eta^5 & \sqrt{2}\eta^6 \\ \sqrt{2} & \eta^2 + \eta^4 & 1 + \eta^3 & \eta^2 + \eta^3 \\ \sqrt{2} & \eta + \eta^5 & \eta + \eta^2 & 1 + \eta^5 \\ \sqrt{2} & 1 + \eta^6 & \eta^4 + \eta^6 & \eta + \eta^4 \end{pmatrix} \quad (6.17)$$

These matrices satisfy the relations for $SL_2(7)$ which are

$$A^{[4]^2} = B^{[4]^3} = (A^{[4]}B^{[4]})^7 = [A^{[4]}, B^{[4]}]^4 = -I$$

The generators in the triplet representation are the following

$$A^{[3]} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^2 - \eta^5 & \eta^6 - \eta & \eta^3 - \eta^4 \\ \eta^6 - \eta & \eta^4 - \eta^3 & \eta^2 - \eta^5 \\ \eta^3 - \eta^4 & \eta^2 - \eta^5 & \eta - \eta^6 \end{pmatrix} \quad (6.18)$$

and

$$B^{[3]} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta - \eta^4 & \eta^4 - \eta^6 & \eta^6 - 1 \\ \eta^5 - 1 & \eta^2 - \eta & \eta^5 - \eta \\ \eta^2 - \eta^3 & 1 - \eta^3 & \eta^4 - \eta^2 \end{pmatrix} \quad (6.19)$$

As expected, the $A^{[3]}$ and $B^{[3]}$ satisfy the defining relations

$$A^{[3]2} = B^{[3]3} = (A^{[3]}B^{[3]})^7 = [A^{[3]}, B^{[3]}]^4 = I \quad (6.20)$$

We note that our representations $A^{[3]}, B^{[3]}$ are connected to the conjugate triplet of those of refs [17, 26] through the similarity transformation obtained by the diagonal matrix $(1, -1, -1)$. We also note in passing that the phenomenological implications of $PSL_2(7)$ have been analysed in several works (see reviews [1, 2, 3] and references therein).

7 Conclusions

In the present note, we have introduced an intriguing relation of the discrete flavour symmetries with the automorphisms of the magnetic translations of the finite and discrete Heisenberg Group. This relation is reminiscent of the discrete symmetries of the Quantum Hall effect, where in a toroidal two dimensional space the magnetic flux transforms the torus to a phase space and the Hilbert space of a charged particle becomes finite dimensional and the corresponding torus effectively discrete [27]. Torii with fluxes in internal extra dimensions appear naturally in the framework of F-theory of elliptic fibrations over Calabi-Yau manifolds, where they generate the GUT gauge groups and other discrete symmetries at particular singularities of the fibration. Phenomenological explorations have shown that such discrete symmetries are particularly successful in predicting the fermion mass hierarchies and the flavour mixing. Inspired by these observations we made use of the discrete Heisenberg Group to develop a simple and unified method for the derivation of basic non-trivial representations of a large class of non-abelian finite groups relevant to the flavour symmetries. It will be important to construct explicit models of elliptic fibrations with fluxes, where the discrete magnetic translations appear naturally and the discrete flavour symmetries as their automorphisms.

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